$$H^*(BSO(2n);\mathbb{Z}) = \mathbb{Z}[p_1,\dots,p_n,e]\big/\langle e^2=p_n\rangle \oplus \text{Torsion}$$

where Torsion is as above

D. Obstruction Theory, take two

suppose $A \subset X$ and we have a map $f: A \to Y$

Can we extend f to a map $X \to Y$?

Use obstruction theory again!

as usual assume

A) (X,A) a relative CW-complex

(so $X^{(-1)} = A$ and $X^{(k)}$ obtained from $X^{(k-1)}$ by attaching k-cells)

B) Y is n-simple for all n

(i.e. $\pi_1(Y)$ acts trivially on $\pi_n(Y)$

so $\pi_n(Y) = [S^n, Y]$)

Th$^{\underline{m}}$ 19:

$\boxed{\text{given } (X,A) \text{ satisfying A) and } Y \text{ satisfying B)} \text{ and } f: X^{(n)} \to Y \text{ then 1) } \exists \text{ a cocycle } \mathfrak{o}(f) \in C^{n+1}(X,A;\pi_n(Y)) \text{ which vanishes}}$

$\iff$

f extends to $X^{(n+1)}$

2) $\sigma(f) = [\tilde{\sigma}(f)] \in H^{n+1}(X, A; \pi_n(Y))$ vanishes

$\iff$

$f|_{X^{(n-1)}}$ extends to $X^{(n+1)}$

Proof: just like in Section A

$$\tilde{\sigma}(f) : C^{CW}_{n+1}(X, A) \longrightarrow \pi_n(Y)$$

$$\underset{\bigoplus \mathbb{Z} e_i^{n+1}}{\shortparallel}$$

free abelian group generated by $(n+1)$ cells

is defined as follows:

e_i^{n+1} is attached by a map

$\phi_i : (\partial e_i^{n+1} = S^n) \longrightarrow X^{(n)}$

so $\tilde{\sigma}(e_i^{n+1}) = [f \circ \phi_i] \in [S^n, Y] \cong \pi_n(Y)$

exercise:

1) $\tilde{\sigma}(f) = 0 \iff f$ extends to $X^{(n+1)}$

2) $\tilde{\sigma}(f)$ unchanged under homotopy of f

3) $\delta \tilde{\sigma}(f) = 0$

4) given $f, g : X^{(n)} \longrightarrow Y$ that agree on $X^{(n-1)}$

then $\exists\, \tau(f, g) \in C^n(X, A; \pi_n(Y))$ s.t.

$$\delta \tau(f, g) = \tilde{\sigma}(f) - \tilde{\sigma}(g)$$

5) by varying the homotopy class of f on $X^{(n)}$, relative to $X^{(n-1)}$, we can change $\tilde{\sigma}(f)$ by an arbitrary coboundary

<u>Hint</u>: See proofs of Lemmas 1 and 2

Th$^{\underline{m}}$ follows $\blacksquare$

<u>Th$^{\underline{m}}$20:</u>

let $f, g: X \to Y$ be given (satisfying A), B) above) and $H: X^{(n)} \times [0,1] \to Y$ a homotopy $f|_{X^{(n)}}$ to $g|_{X^{(n)}}$ the obstruction to extend H to $X^{(n+1)} \times [0,1] \to Y$ lies in

$$H^n(X, A; \pi_n(Y))$$

<u>Proof</u>: by Th$^{\underline{m}}$19 we get an obstruction in

$$H^{n+1}(X \times [0,1], ((A \times [0,1]) \cup (X \times \{0,1\})); \pi_n(Y))$$

now let $U_1 = X \times [0, 3/4]$, $V_1 = (X \times \{0\}) \cup (A \times [0, 3/4])$

$U_2 = X \times [1/4, 1]$, $V_2 = (X \times \{1\}) \cup (A \times [1/4, 1])$

$U_1 \cap U_2 = X \times [1/4, 3/4]$, $V_1 \cap V_2 = A \times [1/4, 3/4]$

and since (X,A) an NDR pair Lemma I.9 says $(X \times \{0\}) \cup A \times [0, 3/4]$ is a retract of $X \times [0, 3/4]$

so $H^n(U_1, V_1) = 0$

now

$$\underbrace{H^n(U_1,V_1)\oplus H^n(U_2,V_2)}_{\overset{\shortparallel}{0}} \to H^n(U_1\cap U_2, V_1\cap V_2) \to H^{n+1}(U_1\cup U_2, V_1\cup V_2) \to \underbrace{H^{n+1}(U_1,V_1)\oplus H^{n+1}(U_2,V_2)}_{\overset{\shortparallel}{0}}$$

so $H^n(X\times[1/4,3/4], A\times[1/4,3/4]) \cong H^n(X\times[0,1], (X\times\{0,1\})\cup A\times[0,1])$

$$\overset{\cong}{H^n(X,A)}$$

so obstruction lives in claimed group! ▦

Th^m 21:

let (X,A) be a relative CW-complex and Y be an n-simple space for all n

if $\pi_k(Y)=0$ $\forall k<n-1$, then for any $f: A\to Y$ $\exists$ an extension $\tilde{f}: X^{(n)}\to Y$ and the obstruction $[\sigma(\tilde{f})]$ only depends on f

so denote it $\gamma^{n+1}(f)$ ← primary obstruction

moreover if $g:(X',A')\to(X,A)$ then

$$g^*(\gamma^{n+1}(f)) = \gamma^{n+1}(f\circ g)$$

Proof: just like proof of Th^m 4 ▦

Th^m 22 (Brown Representation Th^m):

let (X,A) be a relative CW pair

there is a natural bijection

$$[(X,A),(K(\pi,n),x_0)] \leftrightarrow H^n(X,A;\pi)$$

Elenberg-Mclone space

π abelian

Proof: by Hurewicz $H_k(K(\pi,n))=0$ for $k<n$

and $H_n(K(\pi,n)) \cong \pi_n(K(\pi,n)) = \pi$

the Universal Coefficients Th$^{\underline{m}}$ says

$$H^n(K(\pi,n);\pi) \cong \mathrm{Hom}(H_n(K(\pi,n)),\pi) \oplus \mathrm{Ext}(\underbrace{H_{n-1}(K(\pi,n))}_{\substack{\parallel \\ 0}},\pi)$$

$$\cong \mathrm{Hom}(\pi,\pi)$$

let $\iota \in H^n(K(\pi,n),\pi)$ correspond to $id:\pi \to \pi$

define $\Psi:[(X,A),(K(\pi,n),x_0)] \longrightarrow H^n(X,A;\pi)$

$$f \longmapsto f^*\iota$$

note: since $\pi_k(K(\pi,n))=0 \ \forall k<n$ the first obstruction to homotoping a map $f:(X,A) \to (K(\pi,n),x_0)$ to be constant lives in $H^n((X,A),\pi)$

Claim: this obstruction is $\Psi(f)$

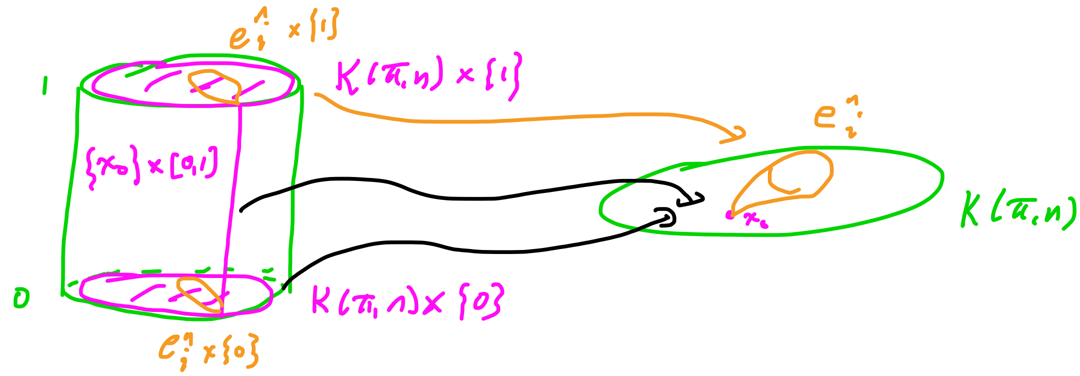
to see this note that by then naturality of the primary obstruction we just need to check that ι is the primary obstruction to homotoping the identity map $K(\pi,n) \to K(\pi,n)$ to the constant map

we know $K(\pi,n)^{(n-1)} = x_0$

so id and constant map agree on $K(\pi,n)^{(n-1)}$

the n-cells e_i^n correspond to generators of $\pi = \pi_n(K(\pi, n))$

$$\tilde{\Theta} \in H^n((K(\pi,n), x_0); \pi) \cong H^{n+1}((K(\pi,n), (\{x_0\}\times[0,1]) \cup (K(\pi,n)\times\{0,1\}))$$



so $\tilde{\theta}$ sends e_i^n to generator corresp e_i^n

i.e. $\tilde{\Theta} = \iota$ so claim is true

Claim: Ψ is onto

let $\alpha \in H^n(X,A;\pi)$

$\exists\ \tilde{\alpha} \in C^n(X,A;\pi)$ s.t. $\alpha = [\tilde{\alpha}]$

so $\tilde{\alpha}: C_n(X,A) \to \pi$

define f_α on to be constant on $X^{(n-1)}$

and for each n-cell e_i^n of X

$$f_\alpha : e_i^n \to K(\pi,n)$$

represents $[f_\alpha(e_i^n)] = \tilde{\alpha}(e_i^n) \in \pi = \pi_n(K(\pi,n))$

this gives f_α on $X^{(n)}$

for each e_j^{n+1} of X note

$$\tilde{\alpha}(\partial e_j^{n+1}) = 0 \quad \text{since } \delta\tilde{\alpha} = 0$$

so $f_\alpha(\partial e_j^{n+1})$ is null-homotopic in $K(\pi,n)$

and we can extend f_α over e_j^{n+1}, ie over $X^{(n+1)}$

but now $\pi_k(K(\pi,n)) = 0 \ \forall k > n$,

so no obstruction to extending f_α to $f_\alpha : X \to K(\pi,n)$

as in proof of first claim we clearly have

$$\Psi(f_\alpha) = f_\alpha^*(\iota) = \alpha$$

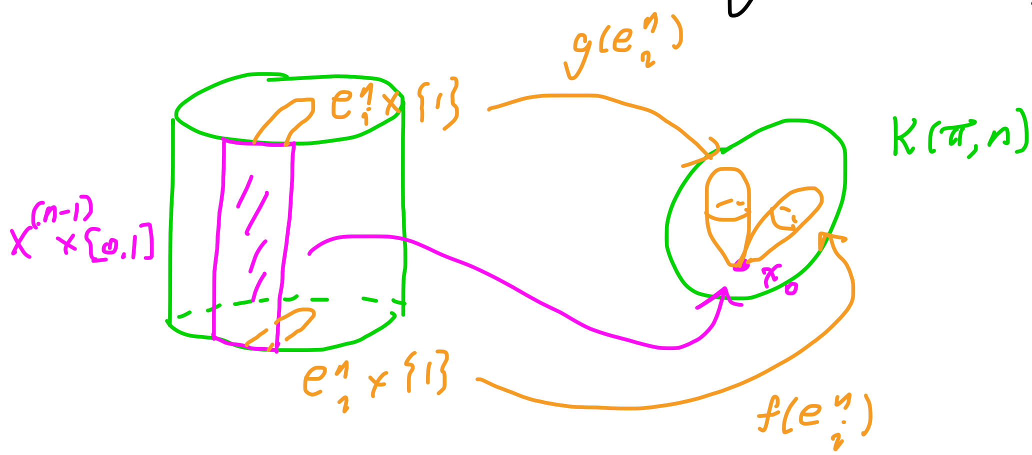
Claim: Ψ is injective

suppose $f,g : (X,A) \to (K(\pi,n), x_0)$ s.t. $\Psi(f) = \Psi(g)$

the primary, and only, obstruction to a homotopy from f to g lives in

$$\theta \in H^n(X,A;\pi)$$

if we evaluate on e_i^n we get $\theta(e_i^n)$



so $\theta(e_i^n) = f(e_i^n) - g(e_i^n) \in \pi_n(\pi,n)$

$$= \iota((f_* - g_*)(e_i^n))$$

$$= (f^*\iota - g^*\iota)(e_i^n) = 0$$

$\therefore f$ is homotopic to g ∎